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EFFICIENT COMPUTATION AND LONG RANGE
OPTIMIZATION APPLICATIONS USING
A TWO-CHARACTERISTIC MARKOV-TYPE
MANPOWER FLOW MODEL

by

Kneale T. Marshall

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ABSTRACT

In [1] the author has compared and contrasted Markov and longitudinal manpower flow models. The Markov model requires relatively little data and has been widely analyzed (see [2] and [3]). The longitudinal model incorporates more realistic personnel flows, but requires extensive data which is not always available. In [4] Hayne and Marshall analyze a two-characteristic Markov model which can be viewed as a hybrid of the Markov and longitudinal models. The purpose of this paper is to show how efficient computational methods can be used with the two-characteristic model by exploiting the special structure of its underlying matrix. These methods make possible the efficient use of this basic flow model in optimization models similar to those described in Chapter 5 of [3].

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1. Introduction

In a previous paper [1] the author has compared and contrasted cross-sectional and longitudinal manpower flow models. The cross-sectional, or Markov, model requires relatively little data and has been widely analyzed (see, for example, [2] and [3]). The longitudinal model incorporates more realistic personnel flows, but requires extensive data which is not always available. In [4] Hayne and Marshall analyze a two-characteristic cross-sectional model which can be viewed as a hybrid of the Markov and longitudinal models. In Chapter 5 of [3] Grinold and the author present some long-range optimization models based primarily on the longitudinal model.

The purpose of this paper is to show how efficient computational methods can be used with the two-characteristic model by exploiting the special structure of its underlying matrix. These methods make possible the efficient use of this basic flow model in optimization models similar to those described in [3]. This paper explores this application in detail when the two characteristics of the state are grade and time-in-grade. Approximate solutions are found to infinite horizon linear programs using methods similar to those in [3].

This paper relies heavily on the notation and results in [3] and [4]. The reader is referred there for details. Section 2 contains a formulation of the optimization model and its approximation (see [3], Chapter 5, pages 186-209). Section 3 contains results on the structure of the flow matrices, their generating function for the (Grade, Time in Grade) model, and the legacies and their generating function. Section 4 describes efficient methods of computation for these generating functions. Section 5 gives a simple numeric example which the reader may wish to follow simultaneously with the theory in Sections 3 and 4. The example has been kept small and simple because of space limitations and for ease of exposition.

2. Model Formulation

It is assumed that manpower joins a system on one of K chains, and at some discrete time t is counted in one of n classes if it is still in the system. Let $P(u)$ be an $n \times K$ matrix with element $p_{ij}(u)$ equal to the fraction of manpower that enters on chain j which is in class i, u periods after entering the system. The matrices $P(0)$, $P(1)$, ... describe the flow through the system.

Let $g(u)$ be a K-element column vector of flows into the system on each chain at time $u = 1,2,\ldots$; let $\ell(u)$ be an n-element column vector of legacies in each class at time $u = 1,2,\ldots$ of manpower which enter up to and including time $t = 0$, which is taken to be the current time. If $s(t)$ is an n-element vector of stocks in each class at time $t = 1,2,\ldots$, then

(1) $$s(t) = \sum_{u=1}^{t} P(t-u)\; g(u) + \ell(t)\ .$$

In [3] Grinold and Marshall used this chain-flow model as the basis of a linear optimization model (see Chapter 5). Let α be a discount factor, a an n-element vector of one-period costs on stocks, b a K-element of one-period costs on new hires, ρ the constant size of the system, A a constraint matrix on stocks and B a constraint matrix on flows. Finally let e be a vector with all elements equal to 1.

Consider the infinite horizon linear program (LP)

P1.
$$\text{Minimize} \quad \sum_{t=1}^{\infty} \alpha^{t}[as(t) + bg(t)]$$

$$\text{Subject to:} \quad \left.\begin{aligned} es(t) &= \rho \\ As(t) &\geq 0 \\ Bg(t) &\geq 0 \\ g(t) &\geq 0 \end{aligned}\right\} \quad t = 1,2,\ldots$$

It is shown in [3] that the solution to a K-variable single period problem can be used to generate solutions to P1 which are usually optimal and always good approximations. Let

$$\tilde{P}(\alpha) = \sum_{u=0}^{\infty} \alpha^{u} P(u), \qquad \tilde{\ell}(\alpha) = \sum_{u=1}^{\infty} \alpha^{u}\, \ell(u) \quad \text{and} \quad c = a\tilde{P}(\alpha) + b.$$

Then by using (1) to eliminate $s(t)$, and multiplying the t-th period constraints in P1 by α^{t} and summing, we obtain

P2. Minimize cg

Subject to: $e\tilde{P}(\alpha)g = [\alpha\rho/(1-\alpha)] - e\tilde{\ell}(\alpha)$

$$A\tilde{P}(\alpha)g \geq -A\tilde{\ell}(\alpha)$$

$$Bg \geq 0$$

$$g \geq 0$$

This LP has only K variables g. Let g^* be the optimal solution to P2. In [3], Chapter 5, it is shown how scalars $\gamma(1), \gamma(2), \ldots$ can easily be found successively such that, if we let $g^*(t) = \gamma(t)g^*$, then these $g^*(t)$ are often optimal in P1 (they are optimal if A is vacuous).

The purpose of this paper is to analyze the structure of $\tilde{P}(\alpha)$ and $\tilde{\ell}(\alpha)$ in P2 when

(i) The system can be partitioned into a hierarchical structure of grades

(ii) A two-characteristic cross-sectional flow model is used with a <u>state</u> described by the 2-tuple (Grade, Time in Grade). This is called the (G, TIG)-model.

(iii) Entering a chain corresponds to entering a grade, necessarily with TIG equal to 1.

(iv) A class corresponds to a grade.

We use the results of Hayne and Marshall in [4] for the (G,TIG) model. For example, in a simple model of a university faculty the grades might be Assistant Professor, Associate Professor and Full Professor. Let us assume that the maximum number of periods (years) a person can spend in these grades is 6, 30, and 30. The number of states in the (G,TIG) model is 66, so that a cross-sectional flow matrix, say Q, (66×66) would have 4356 elements, most of which would have value zero. One of the matrices $P(u)$ in (1) would be 3×3 with only 9 elements, since chains map onto grades with TIG equal to 1 and stocks are also measured only in grades. Any given manpower policy will imply certain element values for Q. In this paper we show how to find $P(u)$ and $\ell(u)$, and hence $\tilde{P}(\alpha)$ and $\tilde{\ell}(\alpha)$, from Q in an efficient manner. Thus policy changes can be reflected in the chain flow model through Q, where the effects of such changes on $P(u)$ could not be determined directly. A simple numerical example is given in Section 5 following the theory. The reader may wish to follow the example concurrently with the theory.

3. The Structure of $\tilde{\ell}(\alpha)$ and $\tilde{P}(\alpha)$

The number of grades in the system is n $(= K)$. Let the maximum time in grade j be $u(j)$, and let $\ell = \sum_{j=1}^{n} u(j)$. Then ℓ is the number of states in the system. Let Q be the one-period flow matrix for the (G,TIG) model. The u-period flows are given by Q^u. We now relate the $\ell \times \ell$ matrix Q^u to the $n \times n$ matrix $P(u)$.

Let Γ be an $n \times \ell$ matrix where row j has

(i) the first $\sum_{i=1}^{j-1} u(i)$ elements equal to 0,

(ii) the next $u(j)$ elements equal to 1,

(iii) the remaining elements equal to 0.

Let Φ be an $\ell \times n$ matrix where column j has

(i) the $(u(j-1)+1)$-th element equal to 1 (element 1 when $j = 1$)

(ii) all other elements equal to zero.

Then

$$P(u) = \Gamma Q^u \Phi , \qquad u = 0,1,2,\ldots . \tag{2}$$

Recall that $\tilde{P}(\alpha) = \sum_{u=0}^{\infty} \alpha^u P(u)$. Using (2), if we let

$$N(\alpha) = (I - \alpha Q)^{-1} \tag{3}$$

then

$$\tilde{P}(\alpha) = \Gamma N(\alpha) \Phi . \tag{4}$$

Recall also that $\tilde{\ell}(\alpha) = \sum_{u=1}^{\infty} \alpha^u \ell(u)$.

Let σ be the ℓ-vector of stocks of manpower in each <u>state</u> at time $t = 0$. Then

$$\ell(u) = \Gamma Q^u \sigma, \qquad u = 1,2,\ldots$$

and therefore,

$$\tilde{\ell}(\alpha) = \alpha\Gamma(N(\alpha) - I)\sigma . \tag{5}$$

Notice from (4) and (5) that to determine $\tilde{P}(\alpha)$ and $\tilde{\ell}(\alpha)$ we need to compute the $n \times \ell$ matrix $\Gamma N(\alpha)$. But first we look at the structure of $N(\alpha)$, the inverse of the large sparse matrix $(I - \alpha Q)$.

In [4] Hayne and Marshall show that for the (G,TIG) model the cross-sectional flow matrix Q has the structure

$$(6) \qquad Q = \begin{bmatrix} Q_1 & & & & \\ P_1 & Q_2 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & P_{n-1} & Q_n \end{bmatrix}$$

where Q_j is a $u(j) \times u(j)$ matrix and P_j is a $u(j+1) \times u(j)$ matrix. All other submatrices of Q have 0's as elements and are suppressed. In addition each Q_j contains zeros except for the lower diagonal,

$$(7) \quad Q_j = \begin{bmatrix} 0 & \cdots & & & & 0 \\ q_{j1} & 0 & & & & \vdots \\ 0 & q_{j2} & \cdot & & & \vdots \\ \vdots & \cdot & \cdot & \cdot & & \vdots \\ \vdots & & \cdot & \cdot & \cdot & \vdots \\ \vdots & & & \cdot & \cdot & \cdot \\ 0 & \cdots & & 0 & q_{j,u(j)-1} & 0 \end{bmatrix},$$

$$j = 1,2,\ldots, n .$$

Also each P_j contains zeros except for the top row,

$$(8) \qquad P_j = \begin{bmatrix} p_{j1} & p_{j2} & \cdots & p_{j,u(j)} \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad j = 1,2,\ldots,n-1.$$

The inverse of $(I - \alpha Q)$ can be written

$$(9) \qquad N(\alpha) = \begin{bmatrix} N_{11}(\alpha) & & & \\ N_{21}(\alpha) & \ddots & & \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots \\ \vdots & & & \ddots \\ N_{n1}(\alpha) & \cdots & N_{n,n-1}(\alpha) & N_{nn}(\alpha) \end{bmatrix}$$

where

$$N_{jj}(\alpha) = (I - \alpha Q_j)^{-1}$$

and

$$N_{ji} = \alpha^{j-i} N_{jj}(\alpha)(P_{j-1}N_{j-1,j-1}(\alpha)) \cdots (P_i N_{ii}(\alpha))$$

for $j > i$. Thus $N(\alpha)$ is completely determined by the inverses $\{N_{jj}(\alpha),\ j = 1,2,\ldots,n\}$.

Now let

(10) $$n_{ki}(j) = q_{j,k-1} \cdot q_{j,k-2} \cdot \cdots \cdot q_{j,i} \qquad \text{for } k > i,$$

the product of (k-i) nonzero elements of Q_j. Then

(11) $$N_{jj}(\alpha) = \begin{bmatrix} 1 & & & & & \\ \alpha n_{21}(j) & 1 & & & & \\ \alpha^2 n_{31}(j) & \alpha n_{32}(j) & 1 & & & \\ \cdot & & \cdot & \cdot & & \\ \cdot & & & \cdot & \cdot & \\ \cdot & & & & \cdot & \cdot \\ \alpha^{n-1} n_{u(j),1}(j) & & & & \alpha n_{u(j),u(j-1)}(j) & 1 \end{bmatrix}$$

$$i = 1,2,\ldots, n\ ,$$

Thus all the elements of $N_{jj}(\alpha)$ are determined from the partial products in (10), and all the elements of $N(\alpha)$ in (9) are determined by multiplication of these with the vectors forming the top rows of the P_j in (8). However, the matrix $N(\alpha)$ in (9) need never be explicitly determined in order to find $\tilde{P}(\alpha)$ and $\tilde{\ell}(\alpha)$. We make the following observations.

(12)
- (i) $P_j N_{jj}(\alpha)$ is a matrix with nonzero elements only in the top row (i.e., it has the same structure as (8)).
- (ii) If Π is any matrix with the same structure as (8) with as many rows as $N_{jj}(\alpha)$ has columns, then $N_{jj}(\alpha)\Pi$ requires only the first column of $N_{jj}(\alpha)$

Now let

$$r = \mathrm{Max}\{u(j),\ j = 1,2,\ldots,\ n\}\ ,$$

and let $W(\alpha)$ be an $n \times r$ matrix with the j-th row equal to the nonzero elements of Q_j multiplied by α, preceded by a 1, and 0's added to the right as necessary. Thus

$$(13)\ W(\alpha) = \begin{bmatrix} 1 & \alpha q_{11} \alpha q_{12} \cdots \alpha q_{1,u(1)-1}, \cdots\cdots 0, \cdots\cdots\cdots, & 0 \\ 1 & \alpha q_{21} \alpha q_{22} \cdots\cdots\cdots\cdots\cdots\cdots \alpha q_{2,u(2)-1}, 0, \cdots, & 0 \\ \vdots & & \\ 1 & \alpha q_{n1} \alpha q_{n2} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \alpha q_{n,u(n)-1} \end{bmatrix}$$

where it has been assumed that $u(1) < u(2) < u(n) = r$ for clarity. Similarly, let $V(\alpha)$ be an $n \times r$ matrix with the j-th row equal to the nonzero elements of P_j multiplied by α, with 0's added to the right as necessary, for $j \leq n-1$, and row n a row of 0's. Thus

$$(14)\qquad V(\alpha) = \begin{bmatrix} \alpha p_{11}, \ldots, \alpha p_{1u(1)}, \ldots, \quad 0, \cdots\cdots\cdots\cdots, & 0 \\ \alpha p_{21}, \cdots\cdots\cdots\cdots\cdots\cdots, \alpha p_{2,u(2)}, 0, \cdots\cdots, & 0 \\ \vdots & \vdots \\ \alpha p_{n1}, \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & , \alpha p_{n,u(n)} \\ 0, \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, & 0 \end{bmatrix}$$

Finally let S be an $n \times r$ matrix of stocks at time zero, where element s_{ij} gives the stocks in grade i with time in grade equal to j. Thus S is a matrix representation of the vector σ with 0's added where necessary. Let $T(j) = \sum_{i=1}^{j} u(i)$. Then

$$(15) \qquad S = \begin{bmatrix} \sigma_1 & \sigma_2 \cdots\cdots & \sigma_{T(1)} & 0 & \cdots\cdots & 0 \\ \sigma_{T(1)+1} & \sigma_{T(1)+2} & \cdots\cdots & \sigma_{T(2)} & 0 & \cdots 0 \\ \vdots & & & & & \vdots \\ \sigma_{T(n-1)+1} & \sigma_{T(n-1)+2} & \cdots\cdots & \cdots\cdots & \cdots\cdots & \sigma_{T(n)} \end{bmatrix}$$

All the information required to determine $\tilde{\ell}(\alpha)$ and $\tilde{P}(\alpha)$ is contained in $W(\alpha)$, $V(\alpha)$, and S. Both can be found using simple row operations on these compact matrices. In the 3 grade, 66 state example given earlier V, W and S are each 3×30 and contain a total of 270 elements. Storage of the Q matrix to find $(I - \alpha Q)^{-1}$ directly would require 4356 elements, and direct inversion would necessitate inverting a 66 dimensional very sparse matrix. Total storage for both Q and σ would be 4422 elements.

4. Computation of $\tilde{\ell}(\alpha)$ and $\tilde{P}(\alpha)$.

Let v and w be any two n-vectors. We define a vector-valued function $F(v,w)$ on these such that

$$(16) \quad F(v,w) = [(v_1 + w_2 v_2 + \cdots + w_2 w_3 \cdots w_n v_n), (v_2 + w_3 v_3 + \cdots + w_3 w_4 \cdots w_n v_n), \ldots, (v_{n-1} + w_n v_n), v_n] .$$

We can extend this function as follows. If V and W are matrices of the same size, then $F(V,W)$ is a matrix X with the same dimensions as V and W, and if v and w are the j-th rows of V and W, respectively, then (16) gives the j-th row of X. The function F is the key to the efficient calculation of both $\tilde{\ell}(\alpha)$ and $\tilde{P}(\alpha)$, since by using it we can efficiently compute $\Gamma N(\alpha)$.

Let U be an $n \times r$ matrix such that element (i,j) is 1 if element (i,j) of $W(\alpha)$ is positive, and is 0 otherwise, where $W(\alpha)$ is given by (13). Also let

$$\nu_j = eN_{jj}(\alpha) \,, \qquad\qquad j = 1,2,\ldots,n,$$

the column sums of $N_{jj}(\alpha)$ in (11). Thus ν_j is a $u(j)$-vector. Now let $\bar{\nu}_j$ be ν_j followed by $r-u(j)$ zero's. Then

$$F(U,W(\alpha)) = \begin{bmatrix} \bar{\nu}_1 \\ \bar{\nu}_2 \\ \vdots \\ \bar{\nu}_n \end{bmatrix} , \tag{17}$$

an $n \times r$ matrix.

Now let m_j be the top row of the matrix $P_j N_{jj}(\alpha)$, $j = 1,2,\ldots, n-1$, a $u(j)$-vector, and let $\bar{m}_j$ be m_j followed by $r-u(j)$ zero's. Then

$$(18) \qquad F(V(\alpha),W(\alpha)) = \begin{bmatrix} \bar{m}_1 \\ \bar{m}_2 \\ \vdots \\ \bar{m}_{n-1} \\ 0 \end{bmatrix},$$

an $n \times r$ matrix. Equations (17) and (18) contain all the required data for $\Gamma N(\alpha)$. Let v^1 be the first element of any vector v. Then using the results in (11) and (12),

$$(19) \quad \Gamma N(\alpha) = \begin{bmatrix} \nu_1 & & & & \\ \nu_2^1 m_1 & \nu_2 & & & \\ \nu_3^1 m_2^1 m_1 & \nu_3^1 m_2 & \nu_3 & & \\ \vdots & & & \ddots & \\ \nu_n^1 m_{n-1}^1 \cdots m_2^1 m_1 & \cdots\cdots & & & \nu_n \end{bmatrix},$$

an $n \times \ell$ matrix.

Consider now the determination of $\tilde{P}(\alpha) = \Gamma N(\alpha)\Phi$. From (19) and the definition of Φ we see that

$$\tilde{P}(\alpha) = \begin{bmatrix} \nu_1^1 & & & & \\ \nu_2^1 m_1^1 & \nu_2^1 & & & \\ \nu_3^1 m_2^1 m_1^1 & \nu_3^1 m_2^1 & \nu_3^1 & & \\ \vdots & & & \ddots & \\ \nu_n^1 m_{n-1}^1 \cdots m_2^1 m_1^1 & \cdots\cdots & & \cdots & \nu_n^1 \end{bmatrix}, \tag{20}$$

an $n \times n$ matrix. This can be constructed from the first columns of (17) and (18).

To determine $\tilde{\ell}(\alpha)$ we require $\Gamma N(\alpha)$ and σ. From (19) it is easy to see that it can be constructed from (17), (18), and (15).

We now illustrate efficient *APL* functions to compute the various matrices. The reader should consult [5] for details of *APL* programming.

Let *M1* and *M2* be any two matrices of the same dimension. Then the *APL* function

(21)

```
    ∇ R←M1 F M2;X
[1]   R←(÷×\M2+0=M2)×⌽+\⌽M1×X+0=X←×\M2
    ∇
```

calculates F(*M1*,*M1*) as defined by (16). Let *V*, *W* and *S* be *APL* variables for the matrices V(α), W(α) and S in (14), (13), and (15) respectively. Then (17) is given by (*W* > 0) *F W*, and (18) by *V F W*. If we let these be *X* and *Y* respectively, and let N be the number of grades and P be $\tilde{P}(\alpha)$, then

(22)
```
P←((⍳N)∘.≥⍳N)×(⍉(N,N)⍴X[;1])××⍀(×\÷1,Z),[1]⍉(N,N-1)⍴Z←¯1↓Y[;1]
```

Finally, if A is the *APL* variable for α, and L the variable for $\tilde{\ell}(\alpha)$, then from (5), and the arguments above

(23)
```
L←A×((+/X×S)+(P×((⍳N)∘.>⍳N)×(N,N)⍴÷Z,1)+.×+/Y×S)-+/S.
```

5. <u>Numerical Example</u>

Assume we have 3 grades and that the maximum time in each grade is 3, 4 and 5 periods. Thus $n = 3$, $u(1) = 3$, $u(2) = 4$, $u(3) = 5$, $r = 5$. The matrix Q in (6) is 12×12 and contains the following 5 non-zero submatrices (see (7) and (8))

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 \\ .9 & 0 & 0 \\ 0 & .8 & 0 \end{bmatrix} \qquad P_1 = \begin{bmatrix} .05 & .1 & .8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ .95 & 0 & 0 & 0 \\ 0 & .9 & 0 & 0 \\ 0 & 0 & .85 & 0 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0 & 0 & .1 & .7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ .9 & 0 & 0 & 0 & 0 \\ 0 & .9 & 0 & 0 & 0 \\ 0 & 0 & .8 & 0 & 0 \\ 0 & 0 & 0 & .7 & 0 \end{bmatrix}$$

The starting stocks are given in the 12-vector

$$\sigma = (100,\ 73,\ 70,\ 82,\ 65,\ 63,\ 58,\ 59,\ 48,\ 30,\ 25,\ 20)\ .$$

Let $\alpha = 0.9$. Then from (13), (14) and (15) we have the 3×5 matrices

$$W(\alpha) = \begin{bmatrix} 1 & .81 & .72 & 0 & 0 \\ 1 & .855 & .81 & .765 & 0 \\ 1 & .81 & .81 & .72 & .63 \end{bmatrix},$$

$$V(\alpha) = \begin{bmatrix} .045 & .09 & .72 & 0 & 0 \\ 0 & 0 & .09 & .63 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} 100 & 73 & 70 & 0 & 0 \\ 82 & 65 & 63 & 58 & 0 \\ 59 & 48 & 30 & 25 & 20 \end{bmatrix}.$$

From (17) and (18), by using (21) we obtain

$$F(U,W(\alpha)) = \begin{bmatrix} 2.393 & 1.72 & 1 & 0 & 0 \\ 3.077 & 2.429 & 1.765 & 1 & 0 \\ 3.236 & 2.761 & 2.174 & 1.63 & 1 \end{bmatrix}$$

and

$$F(V(\alpha),W(\alpha)) = \begin{bmatrix} .538 & .608 & .72 & 0 & 0 \\ .398 & .463 & .472 & .63 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then using (22) and (23)

$$\tilde{P}(\alpha) = \begin{bmatrix} 2.393 & 0 & 0 \\ 1.655 & 3.077 & 0 \\ 0.689 & 1.282 & 3.236 \end{bmatrix}$$

and

$$\tilde{\ell}(\alpha) = (172.7 \quad 691.9 \quad 805.8)\ .$$

Once $\tilde{P}(\alpha)$ and $\tilde{\ell}(\alpha)$ have been determined they can be used in the optimization problem P2 to find optimum input flows each period. Suppose we wish to measure the effect of a change of policy. Let us assume for example that we want to test the effect of limiting the maximum time in grade 2 to 3 periods, and of extending the maximum time in grade 3 to 6 periods. Then $W(\alpha)$, $V(\alpha)$, and S each become 3×6 matrices. But (21) can be used with any sized matrices and so (22) and (23) readily give the new values of $\tilde{P}(\alpha)$ and $\tilde{\ell}(\alpha)$ for use in resolving P2. The computational speeds involved in these calculations, including solving P2 and determining $g^*(t) = \gamma(t)g$, are of the order 1 or 2 seconds for systems with about 10 grades and a maximum time in grade of about 30 periods. Thus interactive computer models can be designed so that the manpower planner can sit at a terminal and test alternate policies or design manpower systems.

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